

# Clearer visibility Hong-Ou-Mandel effect with correlation function based on rates rather than intensities

Krzysztof Rosolek<sup>1</sup>, Kamil Kostrzewa<sup>1</sup>, Arijit Dutta<sup>1</sup>, Wiesław Laskowski<sup>1</sup>, Marcin Wieśniak<sup>1,2</sup>, Marek Żukowski<sup>1</sup>

<sup>1</sup> *Institute of Theoretical Physics and Astrophysics,  
University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland,*

<sup>2</sup> *Institute of Informatics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland*

We test ideas put forward e.g in arXiv:1508.02368, which suggest that using rates in quantum optics can lead to better indicators of non-classicality for states of quantum optical fields with undefined photon numbers. By rate we mean the ratio of registered photons in a given detector to the total number of detected photons in all detectors in the experiment. For the Hong-Ou-Mandel effect for parametric down conversion fields, we show that by using two detector correlation functions which are defined in terms of averages of products of measured rates, rather than usual intensities, one can observe non – classical visibilities beyond 1/2 for significantly higher pump rates. At such rates we already have a partially stimulated emission which leads to significant amplitudes for multiple pairs production, still the new approach allows to clearly see the non-classical dip.

*Introduction.* Superposition principle is at the core of quantum mechanics. Quantum entanglement is simply a manifestation of the superposition rule in the composed systems. All quantum interference effects base on the fact of superposition of probability amplitudes of processes that make a contribution to the given event. In 1987 Hong, Ou and Mandel realized one of the most impressive experiments opening new possibilities in modern quantum optics. Their seminal paper [2] is a powerful confirmation of quantum mechanical predictions concerning quantum interference. The general idea of the standard HOM experiment is as follows [1]. Let us consider two photons entering 50 : 50 beam splitter. Let us assume that the single – photon states take the following form

$$|1\rangle_a = \int dk \xi_a(k) a^\dagger(k) |0\rangle_a, \quad (1)$$

$$|1\rangle_b = \int dk' \xi_b(k') b^\dagger(k') |0\rangle_b, \quad (2)$$

where  $\xi_a(k)$  and  $\xi_b(k)$  are certain functions depending on the wave vector  $k$  and normalized in the following sense

$$\int_R dk |\xi_a(k)|^2 = \int_R dk |\xi_b(k)|^2 = 1. \quad (3)$$

The relations (3) are an immediate consequence of the orthonormality of Fock states. The input (before beam splitter) state reads

$$|\psi\rangle_{in} = |1\rangle_a \otimes |1\rangle_b. \quad (4)$$

The output state is defined by unitary transformation  $\mathcal{U}_{BS}$  which describes action of the beam splitter. Let us write appropriate transformation relations for annihilation operators

$$c(k) = \frac{a(k) + b(k)}{\sqrt{2}}, \quad (5)$$

$$d(k) = \frac{a(k) - b(k)}{\sqrt{2}}. \quad (6)$$

One can introduce so called coincidence function

$$G = \langle n_c n_d \rangle, \quad (7)$$

where  $n_c$  and  $n_d$  are the photon number operators for the spacial modes  $c$  and  $d$  respectively. The  $G$  function can be interpreted as a measure of coincidence counts detected by detectors situated behind the beam splitter. Please observe that if one of the channels is empty (lack of photons)  $G = 0$ . By straightforward calculations it can be shown that

$$G = \frac{1}{2} + \frac{1}{2} \int dk dk' \xi_a(k) \xi_a^*(k') \xi_b(k') \xi_b^*(k). \quad (8)$$

Please notice that in the case of equality  $\xi_a = \xi_b$  we obtain  $G = 0$  what we see as so called Hong – Ou – Mandel (HOM) dip i.e. the coincidence function tends to zero.  $G = 0$  means that photons entering beam splitter are completely indistinguishable. Obviously, the wave vector  $k$  is only one of the parameters whereby we can distinguish photons.

The HOM effect can be applied as an verifier of a quality of single – photon sources [3]. By quality we mean that the source produces only single photons and the produced states are pure. If the input states are single – photon mixed states the coincidence counts do not approach zero because of interference of photons (with some probability) which are distinguishable. The appearance of any non single – photon components will be manifested in nonzero coincidences as well. Obviously nonzero dip without photon number resolution does not inform us what is the reason of clicks in both detectors i.e. multi – photon components, impurity of inputs or both.

The other application of the HOM effect is quantum – optical coherence tomography [4] where HOM interferometer is the main part of the experimental set – up.

The above description of the HOM interference based on single photon states. It turns out that the effect can be demonstrated for  $n$  – photon states and bright macroscopic superpositions as well [5, 6].

In [5] Sanaka et al. presented the Fock state filter based on HOM – type multiphoton interference. They consider

an unbalanced beam splitter (with the reflectivity coefficient  $R$ ) and the product of one – photon state and the  $n$  – photon state (in general a superposition of number state) as the input. For the certain value of  $R$  (depending on  $n$ ) the probability of detecting only one photon in a certain detector  $D$  equals zero. It means that by linearity if the detector  $D$  register one photon we can eliminate appropriate  $n$  – photon component from the Fock state superposition entering beam splitter. Obviously for  $n = 1$  and balanced beam splitter their reasoning reduces to the original HOM experiment.

In [6] authors consider a (very) bright squeezed vacuum state generated via frequency-degenerate type-II parametric down conversion (PDC). Bright (nearly  $10^6$  photons per mode on average) twin beams interfere on a balanced beam splitter. Instead standard coincidence function the observable chosen to measure the effect is the variance of the photon number difference at the outputs normalized by the average of the total number of photons. The reported visibility was 0.999. However this is a measure which is not directly related to the standard definition of interferometric visibility, which in the case of the experiment [6] was a tiny 0.022.

In this paper we present a new approach to data analysis for the HOM interference measurement, based on the tricks introduced in [7, 8]. In [7] Zukowski et al. present the new approach to Bell inequalities for quantum optical fields. The new concept is to replace averages of intensities by averages of rates i.e. the normalized intensities. It leads to removal of some loopholes inherent in earlier approaches to quantum optical Bell inequalities [9]. In [8] the same authors introduce some modification of the standard Stokes parameters. The modified Stokes operators are normalized and the vacuum component is removed. The replacement is as follows:

$$S_\theta = a_\theta^\dagger a_\theta - a_{\theta^\perp}^\dagger a_{\theta^\perp} \rightarrow \hat{\Pi}_{NV} \frac{a_\theta^\dagger a_\theta - a_{\theta^\perp}^\dagger a_{\theta^\perp}}{\hat{N}} \hat{\Pi}_{NV}, \quad (9)$$

where  $\hat{\Pi}_{NV}$  is the projection operator on non – vacuum sectors i.e.  $\hat{\Pi}_{NV} = \hat{1} - |\Omega\rangle\langle\Omega|$  and  $\theta, \theta^\perp$  denote any pair two orthogonal polarizations (generally elliptic), while in the denominator we have the operator for the total number of photons  $\hat{N} = a_\theta^\dagger a_\theta + a_{\theta^\perp}^\dagger a_{\theta^\perp}$ , which is independent of  $\theta$ . Obviously the operation of removing vacuum components is not unitary and all new averages should be properly normalized. This new definition allows to construct improved entanglement criteria for multi-photon states. The new approach allows a better description of polarization correlations of the quantum light.

The aim of this paper is to apply the ideas of [7, 8] to more general situations, as the working example we choose the HOM dip. The state under consideration is two mode bright squeezed vacuum. To be able to compare the depth of a dip, we introduce a mode mismatch parametrized by angle  $\alpha$ . In all consideration we assume that the detectors are photon-counting with perfect efficiency.

The paper is organized as follows. In the first section we discuss the case of classical field beams. In the second section we move to describing the both quantum cases (with old and redefined coincidence functions).

## I. CLASSICAL CASE

It is well known that the maximal visibility of HOM dip in the case of classical fields is 50%. The customarily used definition of visibility for HOM dip is

$$V = \frac{G_{max} - G_{min}}{G_{max}},$$

where  $G = \langle I_a(t) I_b(t) \rangle$ . The subscripts  $a$  and  $b$  denote the exit ports of the beam splitter.  $I_x(t)$  is the instantaneous intensity at exit  $x = a, b$ . The average is over a stochastic mixture of classical fields (this includes random relative phases, which average out first order interference behind the beam splitter). The maximum and minimum are taken with respect of some variation of the conditions of the fields entering the beam splitter (like the relative time shift, polarizations, etc.). As we are not able to measure instantaneous values of the intensities in reality we have integrations over detectors' time resolution  $G = \langle \int dt I_a(t) \int dt' I_b(t') \rangle$ . We have taken the integration (effectively) over full time range, because we are interested in pulsed radiation, and we assume that the pulses' time width is much narrower than the time resolution of the detectors. The redefined correlation function in the classical case, for the same physical conditions reads

$$C = \left\langle \frac{\int dt I_a(t) \int dt' I_b(t')}{I_{tot}^2} \right\rangle,$$

where  $I_{tot} = \int dt I_a(t) + \int dt' I_b(t')$ , and it does not depend on the manipulations on the beams before they enter the beam splitter (under the mentioned relations between the pulses and time resolution of detection). Thus if we write the initial field amplitudes as  $A_1(t)$ , and  $A_2(t)$ , respectively for inputs 1 and 2 of the beam splitter, then we see that  $C$  is just  $G$  for fields of changed amplitudes, namely  $A_1(t)/I_{tot}$ , and  $A_2(t)/I_{tot}$ . As  $V_{max} = 50\%$  is a maximum visibility for  $G$  for any inputs, the maximum visibility for  $C$  cannot breach this value.

## II. QUANTUM CASE

In the previous section we considered HOM interference in terms of classical electric fields. The results showed, that new definition of correlation function ( $G$  and  $C$ ) does not change final visibility of whole process and the maximal value of visibility is  $\frac{1}{2}$ .

Now, let us consider the same process in terms of optical fields, what is a natural consequence of [7] and [8]. Let

us start with the state produced by type I frequency degenerate PDC process. The simplified interaction Hamiltonian is of the form

$$\mathcal{H} = i\chi a^\dagger b^\dagger + h.c., \quad (10)$$

where  $a$  and  $b$  are annihilation operators for both spatial modes,  $\chi$  is a coupling constant depending on the nonlinearity of the crystal and the power of the pumping field. If we take the vacuum as an input state we get two mode bright squeezed vacuum in the output

$$|BSV\rangle = \frac{1}{\cosh K} \sum_{n=0}^{\infty} \tanh K^n |n\rangle |n\rangle, \quad (11)$$

where  $K$  is a gain parameter. Each component of this state is composed of  $2n$  photons, distributed equally in  $a$  and  $b$  spatial modes ( $n$  by each mode).

Now we will consider a modified state

$$|BSV\rangle_\alpha = \frac{1}{\cosh K} \sum_{n=0}^{\infty} \frac{1}{n!} \tanh^n K (a_h^\dagger)^n \times (\cos \alpha b_h^\dagger + \sin \alpha b_v^\dagger)^n |0, 0, 0, 0\rangle, \quad (12)$$

where  $h$  and  $v$  denote respectively matched and mismatched modes, and  $\alpha \in [0, \frac{\pi}{2}]$  is a parameter measuring distinguishability in the spatial mode  $b$  with respect to  $a$ . The (relevant) beam splitter transformations act as follows:  $a_h \rightarrow \frac{1}{\sqrt{2}}(c_h + d_h)$ ,  $b_h \rightarrow \frac{1}{\sqrt{2}}(c_h - d_h)$ , and  $b_v \rightarrow \frac{1}{\sqrt{2}}(c_v - d_v)$ , where  $a, b$  and  $c, d$  are annihilation operators for spacial modes before and after the beam splitter respectively. We have for mismatched modes

$$[a_h, a_v^\dagger] = [b_h, b_v^\dagger] = [c_h, c_v^\dagger] = [d_h, d_v^\dagger] = 0. \quad (13)$$

Please notice that for  $\alpha = 0$  we reproduce the two - mode bright squeezed vacuum state. After all calculations corresponding to the beam splitter transformation, given state takes form (see Appendix)

$$\begin{aligned} \mathcal{U}_{BS}|BSV\rangle_\alpha &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^n \sum_{m=0}^{n-l} \sum_{p=0}^l B_n^{k,l,m,p} \\ &\times \sqrt{(2n-k-l-m)!(l-p)!k!p!} \\ &\times |n-k-l-m, l-p, k, p\rangle, \end{aligned} \quad (14)$$

where  $B_n^{k,l,m,p} = \frac{(-1)^{k+p} \tanh^n K \sin^l \alpha \cos^{n-l} \alpha}{n! 2^n \cosh K} \binom{n}{k} \binom{n}{l} \binom{n-l}{m} \binom{l}{p}$ .

Let us define two following expressions

$$G_Q = \langle \psi | n_a n_b | \psi \rangle, \quad (15)$$

and

$$C_Q = \langle \psi | \hat{\Pi}_{NV} \frac{n_a n_b}{(n_a + n_b)^2} \hat{\Pi}_{NV} | \psi \rangle, \quad (16)$$

where  $n_a$  and  $n_b$  are photon number operators in spatial mode (or detectors)  $a$  and  $b$  respectively. Function  $G_Q$  follows the traditional approach and is a quantum version of  $G$ . The new approach is represented by function  $C_Q$  -

here we divide coincidence number by normalizing factor before we calculate the average values. Please notice that in both cases, if the number of photons in one spatial mode is equal to 0, the product  $n_a n_b$  is equal to 0 as well - no coincidences are observed. The correlation function  $C_Q$  can be rewritten as

$$\begin{aligned} C_Q &= \langle \psi | \hat{\Pi}_{NV} \frac{n_a}{n_a + n_b} \frac{n_b}{n_a + n_b} \hat{\Pi}_{NV} | \psi \rangle \\ &= \langle \psi | \hat{\Pi}_{NV} \frac{n_a}{\hat{N}} \frac{n_b}{\hat{N}} \hat{\Pi}_{NV} | \psi \rangle, \end{aligned} \quad (17)$$

where  $\hat{N}$  is the operator of the total number of photons. The definition is fully consistent with the approach of [7, 8] and formula (9).

There are two special cases, namely  $\alpha = 0$  and  $\alpha = \frac{\pi}{2}$  where we are able to perform accurate analysis (without cutting sum over  $n$ ). As was pointed out for  $\alpha = 0$  the state (13) is just two - mode bright squeezed vacuum. The beam splitter transforms it into following state

$$\begin{aligned} \mathcal{U}_{BS}|BSV\rangle_{\alpha=0} &= \sum_{n=0}^{\infty} \sum_{k=0}^n A_n \binom{n}{k} \\ &\sqrt{(2k)!(2(n-k))!} |2k, 2(n-k)\rangle. \end{aligned} \quad (18)$$

It leads to the analytic expressions for  $G_Q$  and  $C_Q$ . Namely

$$G_Q(\alpha = 0) = \frac{1}{2} \sinh^4 K \operatorname{sech} 2K, \quad (19)$$

$$\begin{aligned} C_Q(\alpha = 0) &= \frac{1}{8} (\tanh^2 K \\ &+ \operatorname{sech}^2 K \ln(\operatorname{sech}^2 K)). \end{aligned} \quad (20)$$

Similarly one can obtain results for  $\alpha = \frac{\pi}{2}$ . After beam splitter the state reads

$$\begin{aligned} \mathcal{U}_{BS}|BSV\rangle_{\alpha=\frac{\pi}{2}} &= \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^n A_n \binom{n}{k} \binom{n}{l} \\ &\sqrt{k!(n-k)!l!(n-l)!} |k, l, n-k, n-l\rangle. \end{aligned} \quad (21)$$

The functions  $G_Q$  and  $C_Q$  are equal to

$$G_Q(\alpha = \frac{\pi}{2}) = \frac{1}{4} (2 - \operatorname{sech} 2K) \sinh^2 K, \quad (22)$$

$$\begin{aligned} C_Q(\alpha = \frac{\pi}{2}) &= \frac{1}{8} (2 \tanh^2 K \\ &- \operatorname{sech}^2 K \ln(\operatorname{sech}^2 K)). \end{aligned} \quad (24)$$

With all that we can calculate the visibilities of the HOM experiment for  $|BSV\rangle_\alpha$  state. One can identify maximal number of coincidence as value of  $G$  functions for  $\alpha = \frac{\pi}{2}$  and minimal value for  $\alpha = 0$ .

$$V_f(K) = \frac{f(\alpha = \frac{\pi}{2}) - f(\alpha = 0)}{f(\alpha = \frac{\pi}{2})}, \quad (25)$$

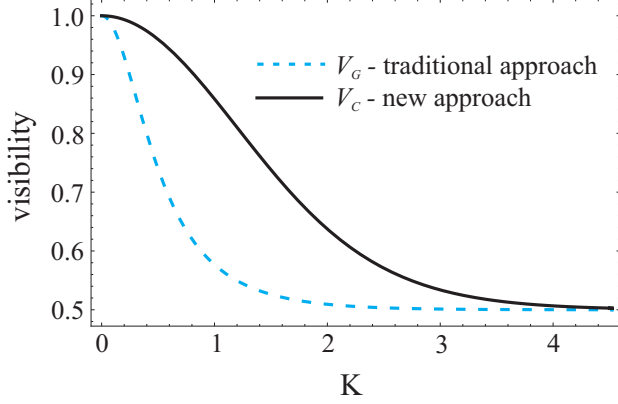


FIG. 1. The visibilities versus parametric gain  $K$ .

where  $f = G_Q, C_Q$ . Using expressions (19), (20), (22), (24) one obtains

$$V_{G_Q}(K) = \frac{1}{2 - \text{sech} 2K} \quad (26)$$

and

$$V_{C_Q}(K) = \frac{3 \sinh^2 K}{-\cosh 2K + \ln(\text{sech}^2 K) + 1} + 2. \quad (27)$$

Fig.(1) shows the visibilities  $V_{G_Q}$  and  $V_{C_Q}$  with re-

spect to the parametric gain  $K$ . Please notice that in the limit of infinite parametric gain  $K$  the difference  $\Delta V := V_{G_Q} - V_{C_Q}$  tends to zero. One can also observe, that both quantum visibilities are greater and tend asymptotically to the classical maximum value  $\frac{1}{2}$ .

### III. CONCLUSIONS

The above considerations show that one can extend the applicability of the ideas of Ref. [8] to optical phenomena which are not polarization based, or related to pairs of spatially separated (multiport) interferometers [10]. The approach with rates leads to a clearer visibility of the Hong-Ou-Mandel effects, for higher pump values.

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### Appendix A: 50 : 50 Beam splitter transformation for $|n, n\rangle$ state

As an initial state we take  $|n, n\rangle$  which is two-mode  $2n$ -photon Fock state. We can express this state in terms of creation operators formalism

$$|n, n\rangle = \frac{1}{n!} (a^\dagger b^\dagger)^n |\Omega\rangle, \quad (A1)$$

where  $a^\dagger$  and  $b^\dagger$  stand for creation operators for two spatial modes and  $|\Omega\rangle$  is a vacuum state. Let us denote the 50 : 50 beam splitter transformation by  $\mathcal{U}_{BS}$ . Hence we get

$$\begin{aligned} |\psi\rangle &= \mathcal{U}_{BS}|n, n\rangle = \frac{1}{2^n} \frac{1}{n!} [(c^\dagger + d^\dagger)(c^\dagger - d^\dagger)]^n |\Omega\rangle \quad (A2) \\ &= \frac{1}{2^n} \frac{1}{n!} (c^{\dagger 2} - d^{\dagger 2})^n |\Omega\rangle \\ &= \frac{1}{2^n} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \hat{c}^{\dagger 2(n-k)} \hat{d}^{\dagger 2k} |\Omega\rangle, \end{aligned}$$

where  $c^\dagger$  and  $d^\dagger$  are creation operators for modes after beam splitter. Moreover the operators  $c^\dagger + d^\dagger$  and  $c^\dagger - d^\dagger$

commute. Using the action of an creation operator on vacuum state the state reads

$$|\psi\rangle = \frac{1}{2^n} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} \sqrt{[2(n-k)]!} \sqrt{(2k)!} \quad (\text{A3})$$

$$\times |2(n-k), 2k\rangle$$

Let us notice that there are only two components in the state (A4), which reveal no coincidences, namely for  $k = 0$  and  $k = n$ .

## Appendix B: 50 : 50 Beam splitter transformation for $|BSV\rangle_\alpha$ state with distinguishable photons

We consider the state (13)

$$|BSV\rangle_\alpha = \frac{1}{\cosh K} \sum_{n=0}^{\infty} \frac{1}{n!} \tanh^n K a_h^{\dagger n} \quad (\text{B1})$$

$$\times (\cos \alpha b_h^{\dagger} + \sin \alpha b_v^{\dagger})^n |0, 0, 0, 0\rangle,$$

where  $h$  and  $v$  are horizontal and vertical polarization respectively and  $\alpha \in [0, \frac{\pi}{2}]$  is a parameter which introduces a kind of polarization distinguishability measure in the spatial mode  $b$ . Please notice that for  $\alpha = 0$  we reproduce the two – mode bright squeezed vacuum state. After beam splitter the states takes the following form

$$\mathcal{U}_{BS}|BSV\rangle_\alpha = \sum_{n=0}^{\infty} A_n (c_h^{\dagger} + d_h^{\dagger})^n$$

$$\left( \cos \alpha (c_h^{\dagger} - d_h^{\dagger}) + \sin \alpha (c_v^{\dagger} - d_v^{\dagger}) \right)^n |0, 0, 0, 0\rangle, \quad (\text{B2})$$

where  $A_n = \frac{\tanh^n K}{n! 2^n \cosh K}$ . Using the binomial expansion we get

$$\mathcal{U}_{BS}|BSV\rangle_\alpha = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^n \sum_{m=0}^{n-l} \sum_{p=0}^l B_n^{k,l,m,p}$$

$$(\hat{c}_h^{\dagger})^{2n-k-l-m} (\hat{d}_h^{\dagger})^k (\hat{c}_v^{\dagger})^{l-p} (\hat{d}_v^{\dagger})^p |0, 0, 0, 0\rangle, \quad (\text{B3})$$

where the coefficients  $B_n^{k,l,m,p}$  equals

$$B_n^{k,l,m,p} = \frac{(-1)^{k+p} \tanh^n K \sin^l \alpha \cos^{n-l} \alpha}{n! 2^n \cosh K} \quad (\text{B4})$$

$$\times \binom{n}{k} \binom{n}{l} \binom{n-l}{m} \binom{l}{p}.$$

Acting on vacuum by the creation operators one obtains (15)

$$\mathcal{U}_{BS}|BSV\rangle_\alpha = \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^n \sum_{m=0}^{n-l} \sum_{p=0}^l B_n^{k,l,m,p} \quad (\text{B5})$$

$$\times \sqrt{(2n-k-l-m)!(l-p)!k!p!}$$

$$\times |n-k-l-m, l-p, k, p\rangle.$$

## Appendix C: Calculation of the expressions $G_Q$ and $C_Q$

We present all calculations concerning the expressions (15) and (16)

$$G_Q = \langle \psi | n_a n_b | \psi \rangle$$

$$C_Q = \langle \psi | \frac{n_a n_b}{(n_a + n_b)^2} | \psi \rangle$$

In order to calculate expectation values (15) and (16) we use the fact that  $f(\hat{n})|n\rangle = f(n)|n\rangle$ , where  $f$  is a certain function of a photon number operator. To calculate expressions (15) and (16) for state (13), we should take summation to infinity into account. However for numerical reasons we cut sums to the certain  $N_{max}$  according to the following criterion

$$\sum_{n=0}^{N_{max}} \frac{\tanh^{2n} K}{\cosh^2 K} \geq 0.99, \quad (\text{C1})$$

where  $\tanh^n K / \cosh K$  is the contribution of the  $n$  – th component in the state (11). Let us notice that the smaller  $K$  we take the smaller  $N_{max}$  we need. For instance, for  $K = 0.5$   $N_{max} = 2$  is reasonable. After rather technical calculation we get

$$C_Q = \sum_{n,n'=0}^{\infty} \sum_{k,k'=0}^{n,n'} \sum_{l,l'=0}^{n,n'} \sum_{m,m'=0}^{n-l,n'-l'} \sum_{p,p'=0}^{l,l'} \quad (\text{C2})$$

$$B_n^{k,l,m,p} B_{n'}^{k',l',m',p'} B_n^{k,l,m,p}$$

$$\frac{(k+m+p)(2n-k-m-p)}{4n^2}$$

$$\Delta(n, k, l, m, p, n', k', l', m', p')$$

$$C_Q = \sum_{\sigma, \sigma'} \xi(\sigma, \sigma') \frac{A}{B}, \quad (\text{C3})$$

where  $\Delta$  is a product of four Kronecker deltas. Namely,

$$\Delta(n, k, l, m, p, n', k', l', m', p') = \delta(k+m, k'+m') \quad (\text{C4})$$

$$\delta(n-k+l-m, n'-k'+l'-m')$$

$$\delta(p, p') \delta(n-l-p, n'-l'-p').$$

For compactness we introduced simplified shortcut notation,  $A = (k+m+p)(2n-k-m-p)$ ,  $B = 4n^2$ .  $\sigma$  and  $\sigma'$  are appropriate multi-indices,  $\xi(\sigma, \sigma')$  contains the rest of expression (C3). Calculating of the expectation value  $G_Q$  we follow the same way. In our shortcut notation one obtains

$$G_Q = \sum_{\sigma, \sigma'} \xi(\sigma, \sigma') A \quad (\text{C5})$$